Black hole interiors in General Relativity.

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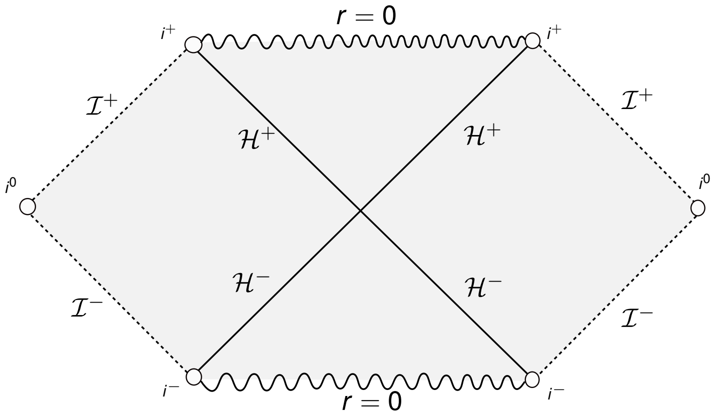
Schwarzschild



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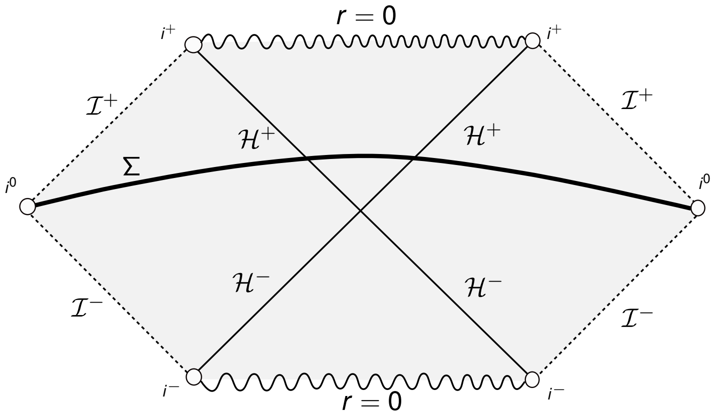

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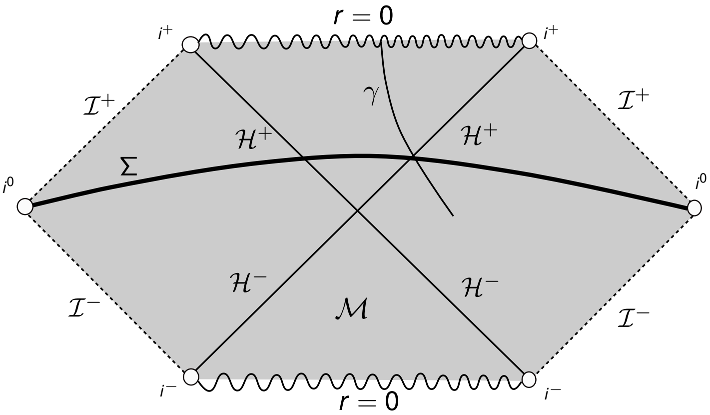
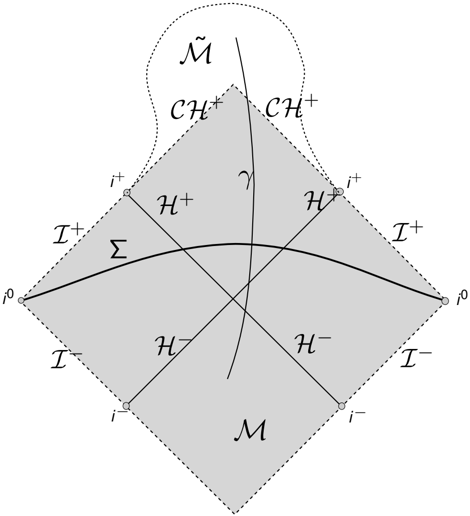
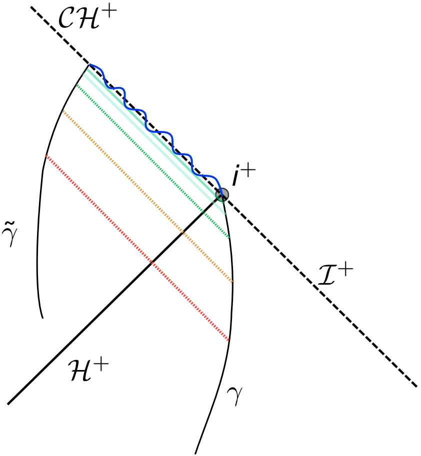


Reissner-Nordström and Kerr



The blue-shift effect ($\Lambda = 0$).
Penrose (1968).



How singular should a terminal boundary be?

- Curvature singularity - inextendibility with metric in C^2
  - Pros: Fairly natural and seems to hold in considerable generality.
  - Cons: it implies neither the breakdown of the field equations nor the destruction of macroscopic observers.

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- Crushing singularity - inextendibility with metric in C^0
  - Pros: Unequivocal terminal boundary. Holds for Schwarzschild (Sbierski 2015) and more generally for the Einstein-scalar field system in spherical symmetry (Christodoulou 1986-1999).
  - Cons: Fails in relevant situations (Dafermos 05, JLC-Girão-Natário-Silva 18, Dafermos-Luk 18)

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- **Einstein singularity - inextendibility with metric in H^1**
  - **Pros: Guarantees the breakdown of the field equations.**

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SCC/Global Uniqueness:

The maximal (globally hyperbolic) development of “generic” compact or asymptotically flat Cauchy initial data for “reasonable” Einstein-matter systems, with a non-negative cosmological constant, is inextendible with metric in H^1.

For the propose of this talk we may simply replace “reasonable Einstein-matter system” with Einstein-Maxwell-scalar field.

- $\phi =$ scalar perturbation of the metric.
- $v =$ affine parameter along the event horizon.
- $\hat{v} =$ outgoing coordinate regular across the Cauchy horizon.
- $\kappa_- =$ surface gravity of Cauchy horizon.
- Price's Law: for some $p > 1$,

$$\partial_v \phi_{|\mathcal{H}^+} \sim v^{-p} \, .$$

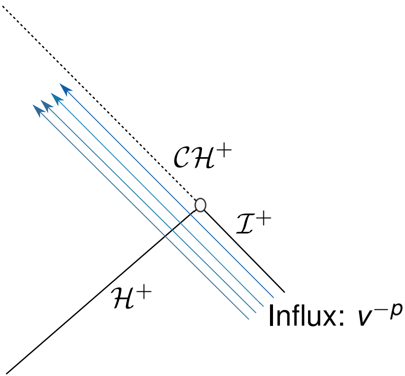
- Rough heuristic:

$$\partial_{\hat{v}} \phi_{|u=u_{int}} \sim v^{-p} e^{\kappa_- v} \to \infty \, .$$

Obs: In the context of the H^1 formulation of SCC the most relevant quantity is in fact

$$\int_{u=u_{int}} (\partial_{\hat{v}} \phi)^2 \, d\hat{v} \, .$$

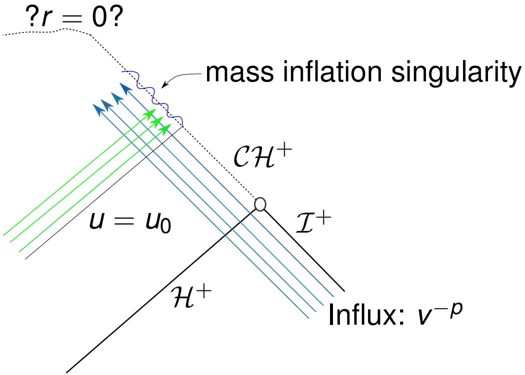
Null dust: the charged Vaidya solution (1951).
Hiscock (1981)



- Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ remains bounded.

Null dust and mass inflation.

Poisson and Israel (1989)

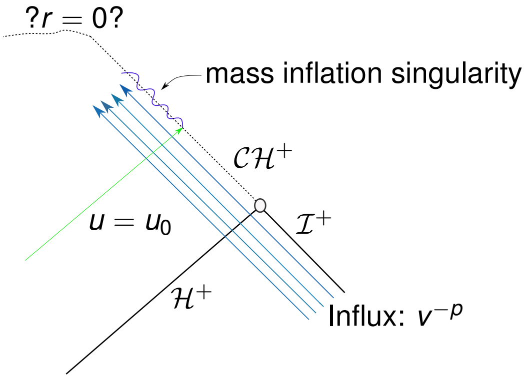


- Renormalized Hawking/Misner-Sharp mass:

$$1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} = |\nabla r|^2 = g_{rr}.$$

- Mass inflation: $\varpi \to \infty$
- Kretschmann scalar: $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \gtrsim \varpi^2 + O(1)$

Null dust and mass inflation.
Ori (1991)



- Tidal forces, although divergent, lead only to finite “tidal deformations”!
- Radius remains bounded away from zero in the onset of the mass inflation singularity.

The Einstein-Maxwell-scalar field (EMS) system

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 2 T_{\mu\nu}$$

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \partial_\alpha \phi \, \partial^\alpha \phi \, g_{\mu\nu} + F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}$$

$$\Box_g \phi = 0$$

$$dF = d^* F = 0$$

- Self-gravitating real massless scalar field:
  - provides a simple matter model with dynamical degrees of freedom in spherical symmetry.
  - exhibits a wavelike behavior reminiscent of the general Einstein vacuum equations.
- A non-trivial Maxwell field is necessary to exclude the Schwarzschild family whose solutions do not contain a Cauchy horizon to start with.

The EMS system, in spherical symmetry ($\Lambda = 0$)

- Price's law:

$$|\partial_v \phi_{|\mathcal{H}^+}| \sim v^{-2+\epsilon} \ . \tag{1}$$

- Assuming (1) holds Dafermos (2003-2012) showed that:
  - A Cauchy horizon (CH) forms.
  - The CH isn't a crushing singularity, at least "initially" (C^0 extensions are allowed).
    This result was later extended to the full vacuum equations by Dafermos and Luk (2018).
  - Nonetheless, the CH is a mass inflation singularity.
  - The CH is expected to be an Einstein singularity (no H^1 extensions allowed).

The EMS system in spherical symmetry ($\Lambda = 0$)

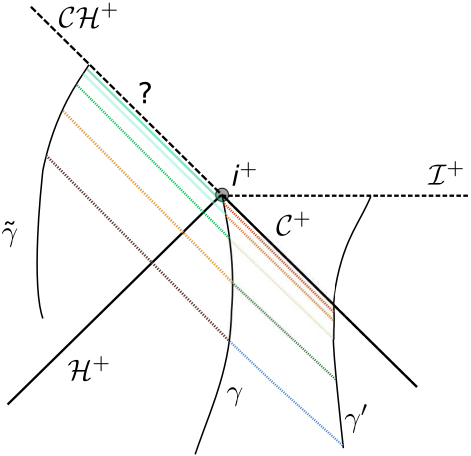
- Price's law:

$$cv^{-2+\epsilon} \lesssim |\partial_v \phi_{|\mathcal{H}^+}| \lesssim Cv^{-2+\epsilon} \,. \quad (2)$$

- Upper bound in (2) established by Dafermos and Rodnianski (2005).
- Lower bound in (2) established by Angelopoulos, Aretakis and Gajic (2018) for the linear wave equation in RN.
- A weaker but more robust integrated version of the lower bound was established by Luk and Oh (2017).

Adding $\Lambda > 0$

The blue-shift effect.



Price Law and Blue-shift revisited

A rough heuristic for Cauchy horizon instability

- Recall Price's Law for $\Lambda = 0$: for some $p > 1$,

$$\partial_v \phi_{|\mathcal{H}^+} \sim v^{-p} \ .$$

- While (a naive) Price's Law for $\Lambda > 0$: for some $\alpha > 0$,

$$\partial_v \phi_{|\mathcal{H}^+} \sim e^{-\alpha v} \ . \tag{3}$$

- For $\Lambda > 0$, the rough heuristic becomes:

$$\partial_{\hat{v}} \phi_{|u=u_{int}} \sim e^{-\alpha v} e^{\kappa_- v} \to ?$$

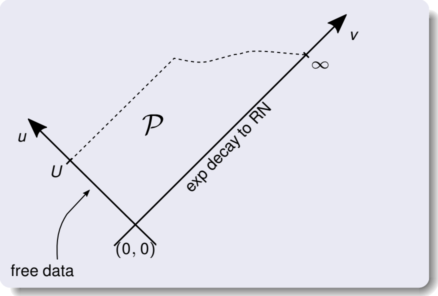
(see for instance Mellor and Moss (1990) and Brady, Moss and Myers (1998))

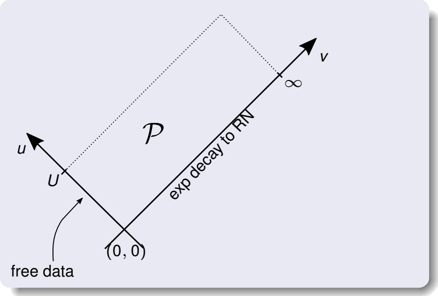
- The 1990s ended with many contradicting claims concerning SCC with $\Lambda > 0$.

The EMS system, in spherical symmetry, with $\Lambda > 0$

JLC-Girão-Natário-Silva (2014-2018)

For the spherically symmetric Einstein-Maxwell-scalar field system with a cosmological constant consider the IVP:





Assume that

$$|\partial_v \phi_{|\mathcal{H}^+}| \sim e^{-\alpha v} \ . \qquad (4)$$

- If $2\alpha < \min\{2\kappa_+, \kappa_-\}$, then mass inflation occurs.
- If $\alpha < \kappa_+$, then either ϖ or $|\partial_{\hat{v}}\phi|$ blow up, which implies that $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded.
- If $\kappa_- < \frac{9}{7}\kappa_+$ and $\alpha > \kappa_+$, or $\frac{7}{9}\kappa_- < \alpha < \kappa_+$, then the mass remains bounded and solutions extend, beyond the Cauchy horizon, with $g \in H^1$ and $\phi \in H^1$.
- In the last case the CH isn't an Einstein singularity.

But what is α?

The linear wave equation in RNdS

Let us take a step back and consider scalar perturbations of the metric, i.e., solutions to the wave equation

$$\Box_g \phi = 0,$$

without symmetries (a step forward) in fixed subextremal RN/Kerr de Sitter backgrounds.

- Dyatlov (2015), following the work of Sá Barreto-Zworsky, Bony-Häfner and Vasy, proved that,

$$|\phi - \phi_0| \lesssim e^{-\alpha\, t^*} \;,$$

  for some $\phi_0 \in \mathbb{C}$.
- Moreover α is given by the spectral gap, i.e., the size of the quasinormal mode (QNM)-free strip below the real axis.

- Hintz and Vasy (2017): Solutions of the wave equation with smooth Cauchy data belong to the Sobolev space

$$H^{\frac{1}{2}+\frac{\alpha}{\kappa_-}}$$

up to, and including, the Cauchy horizon.

- So, introducing the quantity

$$\beta := \frac{\alpha}{\kappa_-}$$

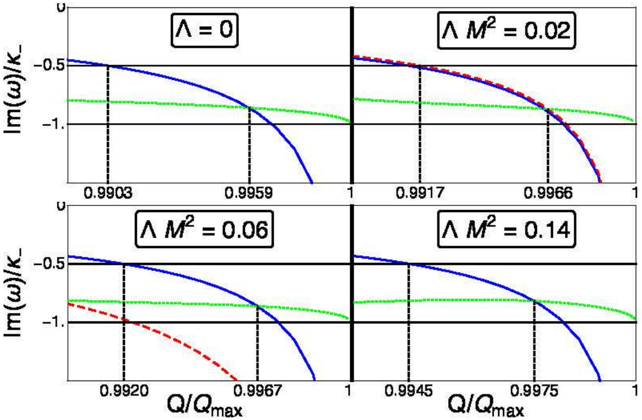
we see that:

  - $\beta < 1/2$ signals an Einstein singularity (H^1- formulation of SCC).
  - $\beta < 1$ is related to blowup of curvature.

We found 3 families of modes

- *Near extremal modes*: $\omega_{0,NE} \approx -i\kappa_-$, dominant for $r_- \approx r_+$.
- *de Sitter modes*: $\omega_{0,dS} \approx -i\sqrt{\frac{\Lambda}{3}}$, dominant for $\Lambda \approx 0$
- *Photon sphere modes*: whose dominant element in the family is provided by the eikonal limit $l \gg 1$ and satisfies $\mathrm{Re}(\omega_{0,PS}) \neq 0$, i.e., it's an oscillating mode.

Some numerical results for neutral fields
Cardoso-JLC-Destounis-Hintz-Jansen (2018)

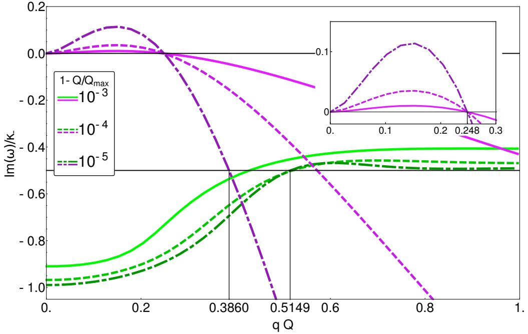


Some numerical results.
Cardoso-JLC-Destounis-Hintz-Jansen (2018)

- Concerning β:
  - $\beta > 1/2$ in the near extremal regime $r_- \approx r_+$, corresponding to near extremal charge.
  - $\beta < 1$ in the entire subextremal parameter range of RNdS.
- This suggests the existence of Cauchy horizons which, upon perturbation, can be seen as singular, by the divergence of curvature, but nonetheless maintain enough regularity as to allow the field equations to determine, in a highly non-unique way, the evolution of gravitation. I.e., the existence of curvature singularities which aren't Einstein singularities.

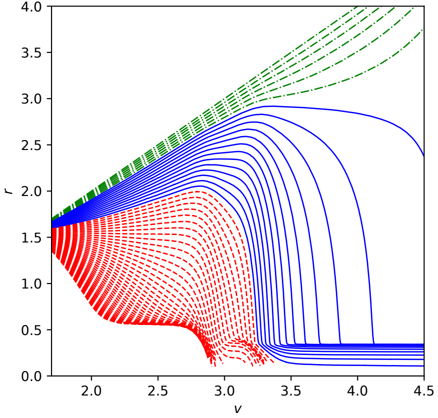
Some numerical results for charged fields

Cardoso-JLC-Destounis-Hintz-Jansen (2018)

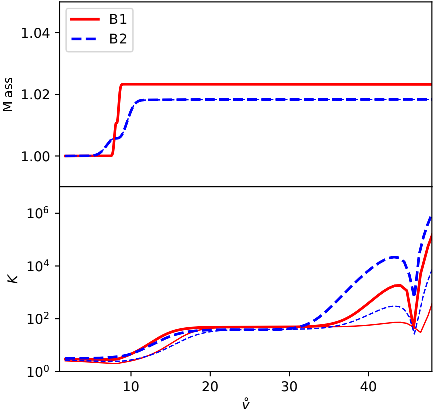


(Compare with Dias et al (2018) and Mo et al (2018))

Some non-linear numerical results for neutral fields
Luna-Zilhão-Cardoso-JLC-Natário (2019)



(Already observed for $\Lambda = 0$ by Burko in 1998)


B1
B2
1.04
1.02
1.00
Mass
10^6
10^4
10^2
10^0
K
10
20
30
40
$\mathring{v}$

What about Kerr de Sitter?

Other numerical results.

- Astrophysical black holes are expected to be nearly neutral. But from the traditional "poor man's" analogy between charge and angular momentum we expected the previous results to tell us something about Kerr-de Sitter.
- Not this time! Dias-Eperon-Reall-Santos (2018) extended the previous QNMs numerical computations to Kerr-de Sitter and found that

$$\beta_{KdS} < 1/2$$

in the entire subextremal parameter range, with

$$\beta_{KdS} \to 1/2$$

as one approches extremality.

- Dafermos and Shlapentokh-Rothmann (2018) constructed solutions to the wave equation in RNdS with $H^{1+\epsilon} \times H^{\epsilon}$ initial data but such that $\phi \notin H^1$ at the Cauchy horizon.
- This suggests that one might try to save SCC by enlarging the allowed set of initial data by weakening their regularity.

In conclusion:

- The presented results suggest a potential failure of SCC in the presence of a positive cosmological constant: there seem to exist large classes of (smooth) Cauchy data leading to a stable Cauchy horizon, beyond which Einstein's equations still make sense.

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- Nonetheless, the expectation changes if we consider neutral spinning de Sitter black holes. Suggesting that an "astrophysical" version of SCC might still hold.
- Quantum effects should also be important since we are generically dealing with unbounded curvature.
    - Recall that for naked singularities "quantum healing" is "the cure".
    - But in the context of SCC one needs some sort of "quantum sickening" (see for instance Hiscock 1980 and Lanir et al 2018).

Ver para crer, em São Tomé e Príncipe

To see is to believe