From Einstein and Eddington to LIGO:
100 years of gravitational light deflection
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From Eddington stars to quasiblack holes and black holes

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Outline

1. Eddington stars

- **First attempts**

·The problem of the gravitational equilibrium of the sun was discussed by Lane (1870), Ritter (1880) who presented polytropic equations and solutions in general form, and Emden with the publication of Gaskugeln (1907).

·The problem of the the origin of the sun's energy source had been addressed by Waterston (1845), Helmholtz (1853), and Kelvin (1880), who advocated the contraction hypothesis.

·The sun and the stars were thought to be composed of a perfect gas of uniform composition, and the energy transport to occur from convection. The exact nature of stellar matter was a debatable question.

·Schwarzschild (1906) brought forward the importance of radiative equilibrium in the solar atmosphere and then Eddington (1916) understood that radiation pressure must stand with gravitation and gas pressure as a third major factor in the maintenance of equilibrium within the whole star, a research culminating in his book The Internal Constitution of the Stars (1926). It opened a new era of stellar structure and evolution along with the possibility of explaining the Hertzprung-Russell diagram.

1. Eddington stars

- **The standard stellar model of Eddington**

·The equations are four: Euler, continuity, energy generation, and heat tranfer,

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}\,, \quad \frac{dm}{dr} = 4\pi r^2\rho\,,$$

$$\frac{dl}{dr} = 4\pi r^2\rho\varepsilon\,, \quad \frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3}\frac{l}{4\pi r^2}\,.$$

They are supplemented by equations of state: $P = P_{\rm gas} + P_{\rm rad} = \frac{R_{\rm gas}}{\mu}\rho T + \frac{1}{3}aT^4$, $\varepsilon = c_{\rm o}\rho^c T^d$, $\kappa = a_{\rm o}\rho^a T^b$. Get P, m, l, and T with boundary conditions $m(0) = 0$, $l(0) = 0$, $P(R) = 0$, and $T(R) = 0$, $m(R) = M$ and $l(R) = L$, the total mass and luminosity. Dividing the fourth by the first yields $\frac{dP_{\rm rad}}{dP} = \frac{\kappa l}{4\pi cGm}$. Dividing the third by the second yields $\frac{dl}{dm} = \varepsilon$. Eddington assumes $\frac{l}{m} = \eta\frac{L}{M}$, with $\eta = 1$ at the surface and increasing inwards. The opacity increases outwards, so assume $\kappa\eta = \kappa_0 =$ constant. Get $\frac{dP_{\rm rad}}{dP} = \frac{\kappa_0 L}{4\pi cGM}$. Integrating

$$P_{\rm rad} = \frac{\kappa_0 L}{4\pi cGM}P\,.$$

Therefore, constancy of $\kappa\eta = \kappa_0$ implies a constant ratio of radiation pressure to total pressure. Define $\beta = \frac{P_{\rm gas}}{P}$, $1-\beta = \frac{P_{\rm rad}}{P}$.

1. Eddington stars

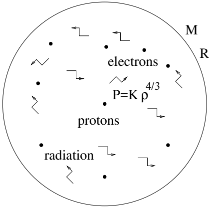
·Then, $P = \frac{P_{\rm gas}}{\beta} = \frac{R_{\rm gas}}{\beta\mu}\rho T$ and $P = \frac{P_{\rm rad}}{1-\beta} = \frac{1}{3}\frac{a}{1-\beta}T^4$.

·Thus

$$P = K\rho^{4/3}, \quad K = \left[\frac{3R_{\rm gas}^4(1-\beta)}{a\mu^4\beta^4}\right]^{1/3},$$

a polytropic equation of state with $n = 3$.

Eddington star



·Use Lane-Emden equation for polytropics results. Put $r = \left(\frac{K}{\pi G\rho_c^{2/3}}\right)^{1/2}\xi$, then $R = \left(\frac{K}{\pi G\rho_c^{2/3}}\right)^{1/2}\xi_3$ and $M = 4\pi c_3\left(\frac{K}{\pi G}\right)^{3/2}$, with $\xi_3 = 6.897$ and $c_3 = 2.018$. Thus $1-\beta = \alpha M^2\mu^4\beta^4$, for $\alpha = \frac{a\pi G^3}{48R_{\rm gas}^4 c_3^2}$. It is a quartic equation for β. When the star's mass M is small one has $\beta = 1$, gas pressure dominates, when M is very large one has $\beta = 0$, radiation pressure dominates.

1. Eddington stars

·Find the central values $\rho_c = \frac{\xi_3^3}{4\pi c_3}\frac{M}{R^3}$, $p_c = \frac{\xi_3^4}{16\pi c_3^2}\frac{M}{R^4}$, $T_c = \frac{\xi_3}{4c_3}\frac{\mu\beta}{R_{\rm g}}\frac{GM}{R}$.

·For the sun get $\beta = 1 - 6.58 \times 10^{-4}$, dominated by gas pressure. Suppose $\mu = 0.6$ then $T_c = 1.2 \times 10^7$ K. A first good estimate for the central temperature. After having found this temperature and believing that a star draws on some vast reservoir of energy unknown at the time, Eddington states (1926) "we do not argue with the critic who urges that the stars are not hot enough for this process; we tell him to go and find a hotter place." Earlier, in a report to the British Association he was prophetic (1920): "If, indeed, the sub-atomic energy in the stars is being freely used to maintain their great furnaces, it seems to bring a little nearer to fullfilment our dream of controlling this latent power for the well-being of the human race - or for its suicide."

·Also can found from the model $L = \frac{4\pi cGM}{\kappa_0}(1-\beta)$. Thus

$$L = \gamma\mu^4\beta(M,\mu)M^3\,,$$

for some γ. The mass-luminosity relation, at the time a theoretical prediction.

2. Compact stars

·A compact star is a star whose radius R is not much larger than its own gravitational radius r_g.

·The first compact star to be observed was a white dwarf. Although relatively large $R \simeq 10^3 r_g$, it baffled astrophysicists in 1910 as its density was amazingly high. It was solved by Fowler (1926) with the degeneracy equation of state, by Anderson (1929) and Stoner (1930) that identified the existence of a mass limit through a relativistic treatment, and finally by Chandrasekhar (1931) for a polytropic equation of state with the definitive mass limit.

·The second compact star to be observed was a neutron star. This is really compact as $R \simeq 2r_g$ and general relativity rather than Newtonian gravitation is needed to explain its structure (Oppenheimer and Volkoff 1939).

·There are no other observed compact stars.

·Compact stars can be realized as solutions of general relativity. Outside, in vacuum, one has, e.g., $ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$, the Schwarzschild solution (1916). Inside one has one form or another of matter, possibly static and spherically symmetric. Here $r_g = 2m$.

2. Compact stars

·Buchdahl bound (1949): Model-independent bound on general relativistic stars stating that for a Schwarzschild exterior the radius R of any nonsingular static perfect fluid body is bounded by $R \geqslant \frac{9}{8} r_{\rm g}$. It is realized by the Schwarzschild interior solution where at the bound the pressure becomes infinite and the spacetime singular. This is a stronger limit than $R \geqslant r_g$ that excludes trapped surfaces.

·Neutron stars obey the Buchdahl bound. They were theoretical constructs for more than thirty years. Since the physicists know that reality goes beyond what meets the eye, it was of great interest to speculate on what was the final state of total gravitational collapse.

·This lead to black holes. Black holes are not compact stars.

3. Black holes

·Black holes were found by Oppenheimer and Snyder (1939). Schwarzschild exterior when joined to matter interior allowed to collapse passing through its own gravitational radius and forming an event horizon indeed yields a black hole. In vacuum r_g is the event horizon r_+, $r_+ = r_g$.

·In its full vacuum form Schwarzschild represents a wormhole, with its two phases, the white hole and the black hole, connecting two asympotically flat universes (Kruskal 1960, Misner-Thorne-Wheeler 1973).

·Other black holes in general relativity: Kerr-Newman family and its particular cases: Schwarzchild, Reissner-Nordström, Kerr. Now a profusion of theoretical BHs of all types, in all theories, with all charges, in all dimensions (Lemos 1997).

·Black holes can be astrophysical, there are many stellar black holes and all galaxies contain a central supermassive black hole, all form from gravitational collapse (Lynden-Bell 1969). We can now detect mergers through gravitational waves (LIGO-Virgo 2015-2019). Could come from physics, small black holes formed from particle collision. Perhaps the BH is the elementary particle of gravitation.

3. Black holes

·Classically, BHs well understood from the outside. For the inside, classically there is no definitive answer, they can harbor singularities, or perhaps they can be regular. The BH interior is one of the outstanding problems in gravitational theory.

·Quantically, BHs still pose problems related to the Hawking radiation and entropy, it is a low energy quantum gravity phenomenon. The singularity itself is a full quantum gravity problem.

·BHs form quite naturally and the uniqueness theorems are quite powerful, but a time immemorial question is: Can there be objects with $R = r_+$? Are there black hole mimickers? (Damour-Solodukhin 2007, Lemos-Zaslavskii 2008).

·It is of great interest to speculate on the existence of compact objects that might obey $R = r_+$. Speculations include gravastars (Mazur-Mottola 2001), highly compact boson stars (Liebling-Palenzuela 2017), and quasiblack holes (Lemos-Zaslavskii 2007-2019).

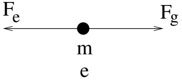
·Here we advocate the QBH. It shows the behavior of highly compact stars.

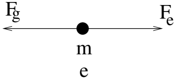
4. Quasiblack holes

- **Examples**

·Putting charge into the matter to bypass the Buchdahl bound a new world of objects and states opens up, objects with $R = r_+$. The charge can be electrical, or angular momentum, or other charge.

·Newtonian gravitation with electric charge, Newton-Coulomb: Two massive charged particles





$F_{\rm g} = \frac{m^2}{r^2}$ and $F_{\rm e} = \frac{e^2}{r^2}$. When $m = e$ $F_{\rm g} = F_{\rm e}$.
Another particle, any number, continuous distribution, any symmetry, any configuration, result holds.

·General relativity, Einstein-Maxwell: Weyl (1917), Majumdar (1947), Papapetrou (1947) showed $ds^2 = -W^2(x^i)\, dt^2 + g_{ij}(x^k)\, dx^i\, dx^j$ for $W^2 = W^2(\phi)$, ϕ the electric potential, in vacuum $W^2 = (\phi + b)^2 + c$. Moreover, for $W^2(\phi) = (\phi + b)^2$ in matter then $\rho_{\rm m} = \rho_{\rm e}$, extremal matter. Called Majumdar-Papapetrou solutions in Hartle-Hawking (1973) when studying a number of extremal BHs scattered around.

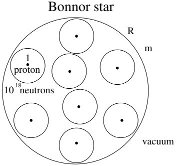
4. Quasiblack holes

·To make a star put a boundary on the matter: Majumdar-Papapetrou makes the interior, exterior is extremal RN, the global solution is a Bonnor star (Bonnor 1953-2000, Lemos-Zanchin 2008).

·Examples of Bonnor stars:

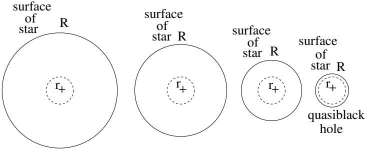
·Star of clouds.

Star has $m = q$.



·Star of supersymmetric stable particles with $m_s = e_s$. Star has total mass m and total charge q with $m = q$.

·For any star radius R the star is in equilibrium. Inclusive for $R = r_+$. What happens when R shrinks to r_+? Something new: a QBH forms.



4. Quasiblack holes

- **Other examples**

·Majumdar-Papapetrou stars asymptotic to extremal Reissner-Nordström (Lemos-Weinberg 2004).

·Bonnor stars with a sharp boundary (Lemos-Zanchin 2008).

·Spheroidal stars made of extremal charged matter (Bonnor 2010).

·Quasiblack holes with pressure: Relativistic charged spheres as the frozen stars (Lemos-Zanchin 2010, de Felice et al 1995).

·Yang-Mills-Higgs magnetic monopoles (Lue-Weinberg 1998, 1999, Lemos-Zanchin 2006).

·Rotating matter at the extremal limit (Bardeen-Wagoner 1971, Lemos-Zas- lavski 2009).

·Matter with spin in Einstein-Cartan theory (Lemos 2019).

·Shells or matter with unbound pressure (Lemos-Zaslavskii-et al 2009-2019).

·Since there are ubiquitous solutions one should consider the core properties of those solutions, the most independently as possible from the matter they are made, in much the same way as one does for BHs.

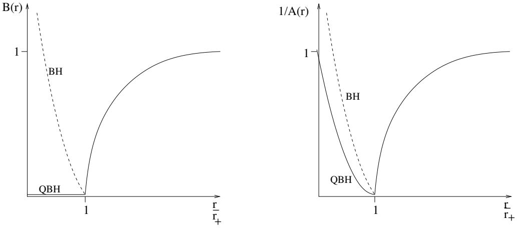
4. Quasiblack holes

- **Features and definition of a QBH**

·Consider a static spherically symmetric metric written as

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\,,$$

Interior metric and asymptotic flat region. Consider the solution has properties
(a) the function $1/A(r)$ attains a minimum at some $r^* \neq 0$, such that
$1/A(r^*) = \varepsilon$, with $\varepsilon \ll 1$, for invariant definition, replace $1/A$ by $(\nabla r)^2$.
(b) For such a small but nonzero ε the configuration is regular everywhere
with a nonvanishing metric function B.
(c) In the limit $\varepsilon \to 0$ the metric coefficient $B \to 0$ for all $r \leq r^*$.



·These three features define a QBH.

4. Quasiblack holes

- **Properties**

·The QBH is on the verge of forming an event horizon, instead, a quasihorizon appears.

·The curvature invariants remain regular everywhere.

·A free-falling observer finds in his frame infinite tidal forces at the interface showing some form of degeneracy, a kind of a singularity.

·Outer and inner regions become mutually impenetrable and disjoint. E.g., in the Lemos-Weinberg solution, interior is Bertotti-Robinson, quasihorizon region is extremal Bertotti-Robinson, and exterior is extremal RN.

·There are infinite redshift whole 3-regions.

·For far away observers the spacetime is indistinguishable from that of black holes. QBHs are black hole mimickers.

·QBHs with finite stresses must be extremal to the outside (Lemos-Zaslavskii 2007, 2008).

·QBHs have continuous radial pressure $p_r^{\text{in}}(r_+) = p_r^{\text{out}}(r_+)$. When extremal from the inside find $p_r(r_+) = -\rho(r_+)$, the same as for dirty BHs.

4. Quasiblack holes

- **Mass formula**

·Put the metric in Gaussian coordinates near the quasihorizon in the form $ds^2 = -N^2dt^2 + dl^2 + g_{ab}dx^a dx^b$, with $a, b = 1, 2$. The Kretschmann scalar is given by $\mathrm{Kr} = P_{ijkl}P^{ijkl} + 4C_{ij}C^{ij}$ where $i, j = 1, 2, 3$. P_{ijkl} is the curvature tensor for $t =$ const, and

$$C_{ij} = \frac{N_{;ij}}{N}\,.$$

$_{;i}$ a covariant derivative. As the metric of the 3-space is positive definite, all terms enter the expression with a positive sign, so each term should be finite separately. QBH means $N = N(x^a) \to 0$. Choose $l = 0$ on the surface. Putting $' \equiv \frac{\partial}{\partial l}$ and $_{;a}$ the covariant derivative for g_{ab}, find

$$\lim_{l\to 0} C_{ll} = \frac{\lim_{l\to 0} N''}{N_0}\,, \quad \lim_{l\to 0} C_{al} = \frac{\lim_{l\to 0} N'_{;a}}{N_0}\,.$$

Finiteness of Kr implies $\lim_{\varepsilon\to 0}\lim_{l\to 0} N'' = 0$ and $\lim_{\varepsilon\to 0}\lim_{l\to 0} N'_{;a} = 0$. Write $N = N_0 + \kappa_1(x^a,\varepsilon)l + \kappa_2(x^a,\varepsilon)\frac{l^2}{2!} + \kappa_3(x^a,\varepsilon)\frac{l^3}{3!} + O(l^4)$ to obtain $\lim_{\varepsilon\to 0}\kappa_1(x^a,\varepsilon) = \kappa$ and $\lim_{\varepsilon\to 0}\kappa_2 = 0$. So at the quasihorizon

$$N = N_0 + \kappa l + \kappa_3(x^a)\frac{l^3}{3!} + O(l^4)\,.$$

4. Quasiblack holes

·Now, when there is matter the mass is given by the Tolman formula (totally different from Komar in the BH case Bardeen-Carter-Hawking 1973),

$$m = \int (-T^0_0 + T^i_i)\sqrt{-g}\, d^3x . \qquad \text{Split} \quad m = M_{\rm in} + M_{\rm surf} + M_{\rm out} .$$

$M_{\rm in} = \int_{\rm in}(-T^0_0 + T^i_i) N\sqrt{g_3} d^3x$, so that $M_{\rm in} \leq N_{\rm B}\,(M_0 + M_k)$. Since $N_{\rm B} \to 0$ one has $M_{\rm in} = 0$. For $M_{\rm out}$ one has $M_{\rm out} = \int_{\rm out}(-T^0_0 + T^k_k) N\sqrt{g_3} d^3x$, so find $M_{\rm out} = \varphi_{\rm h} q + M^{\rm matter}_{\rm out}$.

For the surface, from δ contributions have $S^\nu_\mu = \int T^\nu_\mu\, dl$. So that one gets $M_{\rm surf} = \int (-S^0_0 + S^a_a) N\, d\sigma$. Now, one has $8\pi S^\nu_\mu = \left([K^\nu_\mu] - \delta^\nu_\mu [K]\right)$, where K^ν_μ is the extrinsic curvature tensor, $[...] = [(...)_+ - (...)_-]$, and $+$ and $-$ refer to the outer and inner sides. Find $M_{\rm surf} = \frac{1}{4\pi}\int_{\rm surf}\left[\left(\frac{\partial N}{\partial l}\right)_+ - \left(\frac{\partial N}{\partial l}\right)_-\right] d\sigma$. Now $(\frac{\partial N}{\partial l})_- \to 0$ and $N = N_0 + \kappa l + \kappa_3(x^a)\frac{l^3}{3!} + O(l^4)$, so $(\frac{\partial N}{\partial l})_+ = \kappa$. Finally,

$$M_{\rm surf} = \frac{\kappa A}{4\pi} , \quad A \text{ being the quasihorizon area} .$$

Interpret κ as a surface density (Lemos-Zaslavskii 2018).

4. Quasiblack holes

Putting all the masses together, the total mass of a system containing a quasiblack hole is

$$m = \frac{\kappa A}{4\pi} + \varphi_{\rm h} q + M_{\rm out}^{\rm matter} \,.$$

Same form as the mass formula for black holes and surroundings (BCH 1973), but different means. For vacuum outside, $M_{\rm out}^{\rm matter} = 0$, the mass of the quasiblack hole is $m = \frac{\kappa A}{4\pi} + \varphi_{\rm h} q$, Smarr's formula (Smarr 1973) but for QBHs. For the extremal case $\frac{\kappa A}{4\pi}$ goes to zero and $m = q$. Note: In non-extremal case $M_{\rm surf} \neq 0$ contribution comes from $|S_\nu^\mu| \to \infty$. In extremal case $M_{\rm surf} = 0$ contribution comes from $|T_\nu^\mu|$ finite (Lemos-Zaslavskii 2008).

·When there is rotation $\omega_{\rm h}$ and so there is angular-momentum J the mass formula for quasiblack holes is (Lemos-Zaslavskii 2009)

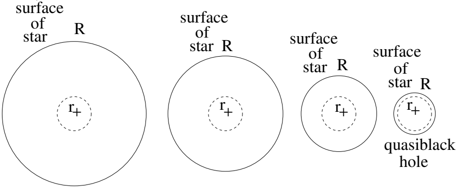
$$m = \frac{\kappa A}{4\pi} + 2\omega_{\rm h} J + \varphi_{\rm h} q + M_{\rm out}^{\rm matter} \,.$$

In vacuum, $M_{\rm \,out} = 0$, $m = \frac{\kappa A}{4\pi} + 2\omega_{\rm h} J + \varphi_{\rm h} q$.

4. Quasiblack holes

- **QBH entropy**

·Imagine a collapsing body. When the surface is at r_+ there is no reason for $S = \frac{1}{4}A_+$, it appears as a jump. By working with a quasistatic collapse and a QBH approach we want to throw some light on the origin of the entropy. We use the 1st law of thermodynamics for the matter and gravitational fields (Lemos-Zaslavskii 2010).



·Generally need an equation of state to integrate the first law of thermodynamics on a path along the energy first and along radius after, say. Not here. Pick another path, namely, choose sequence of configurations such that all members remain on the threshold of horizon formation and integrate over this very subset. The answer should be model-independent. In brief: the approach explores the fact that the boundary almost coincides with quasihorizon.

4. Quasiblack holes

S for spherical configurations:

·Spacetime is composed of some interior, with energy density ρ and pressure p, and the Schwarzschild solution outside, characterized by the mass m or equivalently by the gravitational radius r_+. The surface of the star is at R.

·Consider then a general metric for a static spherically symmetric distribution

$$ds^2 = -N^2(r)\mathrm{e}^{2\psi(r)}dt^2 + \frac{dr^2}{N^2(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

Einstein's equations yield

$$N^2(r) = 1 - \frac{8\pi}{r}\int_0^r d\bar{r}\,\bar{r}^2\rho\,, \quad \psi(r) = 4\pi\int_R^r d\bar{r}\,\frac{(\rho+p)\,\bar{r}}{N^2(\bar{r})}\,,$$

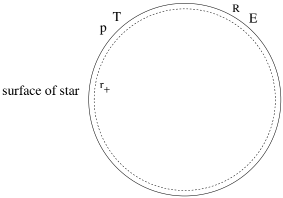
where ρ and p are the matter energy density and pressure, respectively. If the matter is constrained to the region $r \leq R$, then for $r \geq R$ one has $\psi(r) = 0$ and $N^2 = 1 - \frac{r_+}{r}$.

4. Quasiblack holes

·Give the 1st law of thermodynamics in terms of boundary values (York 1986, Brown-York 1993). Since it is spherically symmetric it can be written as

$$TdS = dE + p\,dA\,,$$

where the quantities are locally defined quantities at radius R, say. T is the temperature, S the entropy, E the energy, p the pressure, and A the area.



·Find S by changing simultaneously the radius R and E, to keep it near r_+, with $N \to 0$ for all configurations of interest.

·Write $R = r_+(1+\delta)$ and send $\delta \to 0$ ensuring star is kept near the quasi-horizon. Then integrate the 1st law along such a sequence of quasihorizon objects, counting different members of the same family of states and obtain S for a given r_+.

4. Quasiblack holes

·As an example find the entropy of the simplest configuration, Schwarzschild exterior solution and Minkowski inside, i.e., a thin-shell system at its own gravitational radius. Then, the gravitational radius r_+, the radius of the shell R, and its proper mass M are connected by the relation $r_+ = 2M - \frac{M^2}{R}$. Note $r_+ = 2m$, with m the ADM mass and $E \equiv M$.

Find $M = R(1-N)$, and $p = \frac{(1-N)^2}{16\pi RN}$ the tangential pressure, with $N^2 = 1 - \frac{r_+}{r}$. Use $A = 4\pi R^2$ the area of the shell.

In the process of integrating the 1st law, $TdS = dM + p\,dA$, all three quantities R, r_+, and M, change but in such a way that $R = r_+(1+\delta)$, i.e., change simultaneously R and r_+ when passing from one equilibrium configuration to another. The second term $p = \frac{1}{16\pi RN} \approx \frac{1}{16\pi\, r_+ N}$ is huge since N is small. Also $dA = 8\pi r_+ dr_+$. The first term is $dM \approx dR \approx dr_+$, negligible. Take into account the Tolman formula for the temperature $T = \frac{T_0}{N}$ and that near the quasihorizon the backreaction of quantum fields becomes divergent unless $T_0 = T_{\rm H} = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_+}$ yields

$$S = \frac{1}{4} A_+ \,.$$

4. Quasiblack holes

Interesting that the thin shell offers an alternative route, different from the quasiblack hole approach. It has an exact solution for all shell radii R. Write

$$TdS = dM + pdA\,.$$

Again, take into account the junction to find $16\pi p = \frac{(1-N)^2}{RN}$, with $N^2 = 1 - \frac{r_+}{r}$. Now take into account the integrability conditions, and changing variables from (M,R) to (r_+,R), it turns out that $T = \frac{T_0(r_+)}{N}$ necessarily and the 1st law becomes $dS = \frac{dr_+}{2T_0(r_+)}$, where T_0 has the usual meaning of the temperature measured by an observer at infinity. Hence the entropy can be found by direct integration. Again choose $T_0(r_+) = T_{\mathrm{H}} = \frac{1}{4\pi r_+}$ and find $S = \frac{1}{4}A_+$ (Lemos et al 2008-2019). The formula is valid everywhere including the near-horizon region $R = r_+$.

The entropy of a thin shell does not depend on R. This is a consequence of the fact that there is no matter inside, so $\frac{\partial S}{\partial R} = 0$ everywhere. For general configurations $\frac{\partial S}{\partial R}$ is non-zero, can only find the entropy via the QBH approach.

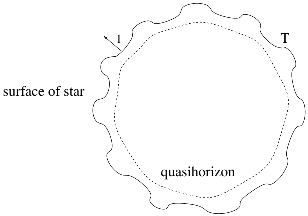
4. Quasiblack holes

S for generic configurations non spherically symmetric:

·In this generic case it is appropriate to use Gauss coordinates. Near the quasihorizon write

$$ds^2 = -N^2dt^2 + dl^2 + g_{ab}dx^a dx^b \,, \quad \text{where} \quad a,b = 1,2.$$

Suppose that the boundary of the compact body is at some $l = \text{constant} = 0$. Local Tolman temperature is T. T_0 is the temperature at asymptotically flat infinity. Relation between them is $T = \frac{T_0}{N}$.



4. Quasiblack holes

· Give the 1st law of thermodynamics in terms of boundary value densities (York 1986, Brown-York 1993)

$$Td(\sqrt{g}\,s) = d(\sqrt{g}\,\varepsilon) + \frac{\Theta^{ab}}{2}\sqrt{g}\,dg_{ab}\,,$$

s is the entropy density, ε is the energy density on the layer defined as $\varepsilon = \frac{K}{8\pi}$. $K = K_{ab}\,g^{ab} = -\frac{1}{\sqrt{g}}\sqrt{g}\,'$. The spatial energy-momentum tensor on the layer Θ_{ab} is $8\pi\Theta_{ab} = K_{ab} + \left(\frac{N'}{N} - K\right) g_{ab}$. These quantities include gravitational as well as matter fields. Can add electric charge.

· Integrate 1st law to obtain S. Usually need an equation of state to integrate on a path along the energy first and along radius after, say. Not here. Pick another path, i.e., choose sequence of configurations such that all members remain on the threshold of horizon formation and integrate over this very subset. The answer should be model-independent.

4. Quasiblack holes

·Now, to have a regular horizon to an outside observer one has $K_{ab} = K^1_{ab} l + O(l^2)$. Then $\varepsilon = K/8\pi$ remains finite. The spatial stresses, $8\pi\Theta_{ab} = K_{ab} + \left(\frac{N'}{N} - K\right) g_{ab}$, diverge due to $\frac{N'}{N}$. In the outer region $N'_+ = \kappa$. The dominant contribution then gives the 1st law in the form $d(s\sqrt{g}) = \frac{\kappa}{16\pi T_0}\sqrt{g} g^{ab} dg_{ab}$. Take into account that near the quasihorizon the backreaction of quantum fields becomes divergent and only the choice $T_0 = T_{\rm H} = \frac{\kappa}{2\pi}$ enables us to obtain a finite result. Thus, $d(s\sqrt{g}) = \frac{1}{4}\, d\sqrt{g}$ and so the entropy density at the quasihorizon is $s\sqrt{g} = \frac{1}{4}\,\sqrt{g}$, up to a constant which we put to zero. Upon integration over the surface, i.e., $\int d^2x$, get

$$S = \frac{1}{4} A_+ \,.$$

This is the Bekenstein-Hawking entropy for a quasiblack hole, or for a black hole through a quasiblack hole approach (Lemos-Zaslavskii 2010, 2018).

·By finding the QBH entropy through a thermodynamic approach intend to throw light on the issue. The modes are on the horizon and seem to be gravitational modes (Lemos-Zaslavskii 2010, 2018).

4. Quasiblack holes

S in the extremal case:

·What about the entropy in the extremal case?

·The entropy of extremal black holes is a particularly interesting problem. Arguments based on periodicity of the Euclidean section of the black hole lead one to assign zero entropy in the extremal case. However, extremal black hole solutions in string theory typically have the conventional value given by the area formula $S = A_+/4$.

·In the extremal case the stresses are finite, and so not all possible modes are excited. This means that the entropy should be $S \leq \frac{1}{4}A_+$. We find using both the quasiblack hole and the thin shell approach that $S = S(A_+)$ with $S(A_+)$ arbitrary. So conclude

$$0 \leq S \leq \frac{1}{4}A_+ \, .$$

It suggests that the extremal entropy depends on the manner the quasiblack hole, and thus the black hole, has formed (Lemos-Zaslavksii 2010, Lemos-Quinta-Zaslavskii 2015-2017, see also Pretorius-Vollick-Israel 1998).

5. Conclusions

·QBH solutions show that there are matter solutions up to the horizon. The Kretschmann scalar is finite, although some degeneracy, a naked horizon.

·It is a kind of membrane paradigm. By taking a timelike matter surface into a null horizon we are recovering the membrane paradigm. A difference is that our membrane is not ficticious like in the paradigm, it is real matter.

·For the entropy the results suggest that the degrees of freedom are on the horizon. It is when a horizon is formed and the system has to settle to the Hawking temperature that the entropy takes the value $A/4$. The results suggest that the degrees of freedom are gravitational modes. When the QBH state is approached the tangential pressure goes to infinite, i.e., to the Planck pressure. Then modes, presumably quantum gravitational modes, are induced.

·Studies are being initiated to understand the entropy of systems with both a black hole and a cosmological horizon (André-Lemos-Zaslavskii 2019).

·Stellar structure started with Eddington 100 years ago. Further improvements on compact stars led Oppenheimer and Snyder in 1939 to show that a black hole was a natural outcome. Have we seen all objects in nature that there are to see? All very difficult but payed for itself, and in what manner! I guess.

6. Published papers

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4) J. P. S. Lemos, O. B. Zaslavskii, 2009, "Angular momentum and mass formulas for rotating stationary quasiblack holes", Physical Review D 79, 044020 (2009).
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6) J. P. S. Lemos, O. B. Zaslavskii, 2010, "Quasiblack holes with pressure: General exact results", Physical Review D 82, 024029 (2010).
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QBHs and self-gravitating thin shells:

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